

THE DIMENSION OF THE SPACE OF \mathbb{R} -PLACES OF CERTAIN RATIONAL FUNCTION FIELDS

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ABSTRACT. We prove that the space $M(K(x, y))$ of \mathbb{R} -places of the field $K(x, y)$ of rational functions of two variables with coefficients in a totally Archimedean field K has covering and integral dimensions $\dim M(K(x, y)) = \dim_{\mathbb{Z}} M(K(x, y)) = 2$ and the cohomological dimension $\dim_G M(K(x, y)) = 1$ for any Abelian 2-divisible coefficient group G .

1. INTRODUCTION

In this paper we study the topological structure and evaluate the topological dimensions of the spaces of \mathbb{R} -places of a field K and of its transcendental extensions $K(x_1, \dots, x_n)$ consisting of rational functions of n variables with coefficients in the field K .

The shortest possible way to introduce \mathbb{R} -places on a field K is to define them as functions $\chi : K \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ to the extended real line, preserving the arithmetic operations in the sense that $\chi(0) = 0$, $\chi(1) = 1$, $\chi(x + y) \in \chi(x) \oplus \chi(y)$ and $\chi(x \cdot y) \in \chi(x) \odot \chi(y)$ for all $x, y \in K$, where \oplus and \odot are multivalued extensions of the addition and multiplication operations from \mathbb{R} to $\bar{\mathbb{R}}$. By definition, for $r, s \in \bar{\mathbb{R}}$, $r \oplus s = \{r + s\}$ if $r + s \in \bar{\mathbb{R}}$ is defined and $r \oplus s = \bar{\mathbb{R}}$ if $r + s$ is not defined, which happens if and only if $r = s = \infty$, in which case $\infty \oplus \infty = \bar{\mathbb{R}}$. By analogy, we define $r \odot s$: it equals the singleton $\{r \cdot s\}$ if $r \cdot s$ is defined and $\bar{\mathbb{R}}$ in the other case, which happens if and only if $\{r, s\} = \{0, \infty\}$.

Historically, \mathbb{R} -places appeared from studying ordered fields. By an *ordered field* we understand a pair (K, P) consisting of a field K and a subset $P \subset K$ called *the positive cone* of (K, P) such that P is an additively closed subgroup of index 2 of the multiplicative group of K . There is a bijective correspondence between positive cones of K and linear orders compatible with addition and multiplication by positive elements. The set $\{a \in K : a > 0\}$ is a positive cone, and the positive cone P generates a total order $<$ on K defined by $x < y \Leftrightarrow y - x \in P$. Each ordered field (K, P) has characteristic zero and hence contains the field \mathbb{Q} of rational numbers as a subfield. This fact allows us to define the *Archimedean part*

$$A_P(K) = \{x \in K : \exists a, b \in \mathbb{Q} : a < x < b\}$$

of the ordered field (K, P) and also to define the canonical \mathbb{R} -place $\chi_P : K \rightarrow \bar{\mathbb{R}}$ on K assigning $\chi_P(x) = \infty$ to each $x \in K \setminus A_P(K)$ and

$$\chi_P(x) = \sup\{a \in \mathbb{Q} : a \leq x\} = \inf\{b \in \mathbb{Q} : b \geq x\} \in \mathbb{R}$$

to each $x \in A_P(K)$. Here the supremum and infimum is taken in the ordered field \mathbb{R} of real numbers.

According to Theorems 1 and 6 of [12], a field K admits a \mathbb{R} -place if and only if it is *orderable* in the sense that it admits a total order. By [3], each \mathbb{R} -place $\chi : K \rightarrow \bar{\mathbb{R}}$ on a field K is generated by a suitable total order P on K .

For an orderable field K denote by $\mathcal{X}(K)$ the space of total orders on K and by $M(K)$ the space of \mathbb{R} -places on K . The mentioned results [12] and [3] imply that the map

$$\lambda : \mathcal{X}(K) \rightarrow M(K), \quad \lambda : P \mapsto \chi_P,$$

assigning to each total order P on K the corresponding \mathbb{R} -place χ_P is surjective.

The spaces $\mathcal{X}(K)$ and $M(K)$ carry natural compact Hausdorff topologies. Namely, $\mathcal{X}(K)$ carries the Harrison topology generated by the subbase consisting of the sets $a^+ = \{P \in \mathcal{X}(K) : a \in P\}$ where $a \in K \setminus \{0\}$. According to [7, 6.1], the space $\mathcal{X}(K)$ endowed with the Harrison topology is

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compact Hausdorff and zero-dimensional. By [4], each compact Hausdorff zero-dimensional space is homeomorphic to the space of orderings $\mathcal{X}(K)$ of some field K .

To introduce a natural topology on the space $M(K)$ of \mathbb{R} -places of a field K , first endow the extended real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with the topology of one-point compactification of the real line \mathbb{R} . It follows from the definition of \mathbb{R} -places that the space $M(K)$ is a closed subspace of the compact Hausdorff space $\bar{\mathbb{R}}^K$ of all functions from K to $\bar{\mathbb{R}}$, endowed with the topology of Tychonoff product of the circles $\bar{\mathbb{R}}$. So, $M(K)$ is a compact Hausdorff space, being a closed subspace of the compact Hausdorff space $\bar{\mathbb{R}}^K$.

It turns out that the topology induced on $M(K)$ by the product topology coincides with the quotient topology induced by the mapping $\lambda : \mathcal{X}(K) \rightarrow M(K)$. This can be seen as follows. By [10], the sets

$$U(a) = \{\chi \in M(K) : \chi(a) \in (0, \infty)\}, \quad a \in K,$$

compose a sub-basis of the quotient topology on $M(K)$. Since those sets are open in the product topology of $M(K)$, the quotient topology is weaker than the product topology. Since the quotient topology is Hausdorff (see [11, Cor.9.9]) and the product topology is compact (so the weakest among Hausdorff topologies), both topologies on $M(K)$ coincide.

The space $M(K) \subset \bar{\mathbb{R}}^K$ is metrizable if the field K is countable. The converse statement is not true as the uncountable field \mathbb{R} has trivial space of \mathbb{R} -places $M(\mathbb{R}) = \{\text{id}\}$. The space of \mathbb{R} -places $M(\mathbb{R}(x))$ of the field $\mathbb{R}(x)$ is homeomorphic to the projective line $\bar{\mathbb{R}}$ while $M(\mathbb{R}(x, y))$ is not metrizable, see [14].

In this paper we shall address the following general problem posed in [2].

Problem 1.1. *Investigate the interplay between algebraic properties of a field K and topological properties of its space of \mathbb{R} -places $M(K)$.*

We shall be mainly interested in the fields $K(x_1, \dots, x_n)$ of rational functions of n variables with coefficients in a subfield $K \subset \mathbb{R}$. It is known that a field K is isomorphic to a subfield of \mathbb{R} if and only if K admits an Archimedean order, i.e., a total ordering P whose Archimedean part $A_P(K)$ coincides with K . This happens if and only if the corresponding \mathbb{R} -place $\chi_P : K \rightarrow \bar{\mathbb{R}}$ is injective if and only if $\chi_P(K) \subset \mathbb{R}$. By $M_A(K)$ we denote the space of injective \mathbb{R} -places on K . Observe that $M_A(K)$ coincides with the space of homomorphisms from K to the real line \mathbb{R} .

A field K will be called *totally Archimedean* if it is orderable and each total order on K is Archimedean. Such fields were introduced and characterized in [16]. Important examples of totally Archimedean fields are the fields \mathbb{Q} and \mathbb{R} . For a totally Archimedean field K the quotient map $\lambda : \mathcal{X}(K) \rightarrow M(K)$ is injective. In this case, the spaces $M(K)$ and $\mathcal{X}(K)$ are homeomorphic and hence the space $M(K)$ is zero-dimensional.

In this paper we shall attack the following:

Conjecture 1.2. *For any subfield $K \subset \mathbb{R}$ and every natural number n the space $M(K(x_1, \dots, x_n))$ of \mathbb{R} -places of the field $K(x_1, \dots, x_n)$ has topological dimension $\dim(K(x_1, \dots, x_n)) \geq n$. If the field K is totally Archimedean, then $\dim(K(x_1, \dots, x_n)) = n$.*

For $n = 1$ this conjecture was confirmed (in a stronger form) in [10]: $\dim M(K(x)) = 1$ for any (also non-Archimedean) real closed field K . The main result of this paper is the following theorem confirming Conjecture 1.2 for $n \leq 2$.

Theorem 1.3. *For any field K admitting an Archimedean order, we get $\dim M(K(x)) \geq 1$ and $\dim M(K(x, y)) \geq 2$. If the field K is totally Archimedean, then $\dim M(K(x)) = 1$ and $\dim M(K(x, y)) = 2$.*

Actually, Theorem 1.3 does not say all the truth about the dimension of the space $M(K(x, y))$. It turns out that this space has covering topological dimension 2 but for any 2-divisible group G the cohomological dimension $\dim_G M(K(x, y))$ is equal to 1! So, the space $M(K(x, y))$ is a natural example of a compact space that is not dimensionally full-valued (which means that the cohomological dimensions of $M(K(x, y))$ for various coefficient groups G do not coincide). A classical example of such a space is the Pontryagin surface, that is a surface with Möbius bands glued at each point of a countable dense subset, see [5, 1.9].

The covering and cohomological dimensions are partial cases of the extension dimension defined as follows, see [6]. We say that the *extension dimension* of a topological space X does not exceed a topological space Y and write $\text{e-dim}(X) \leq Y$ if each continuous map $f : A \rightarrow Y$ defined on a closed

subspace A of X can be extended to a continuous map $\bar{f} : X \rightarrow Y$. The classical Hurewicz-Wallman characterization of the covering dimension [9, 1.9.3] says that $\dim(X) \leq n$ for a separable metric space X if and only if $\text{e-dim}(X) \leq S^n$ where S^n stands for the n -dimensional sphere. The sphere S^n is an example of a Moore space $M(\mathbb{Z}, n)$ (whose reduced homology groups $\tilde{H}_k(S^n)$, $k \neq n$, are trivial except for the n -th group $\tilde{H}_n(S^n)$ which is isomorphic to \mathbb{Z}).

For a non-trivial abelian group G the *cohomological dimension* $\dim_G(X)$ of a compact space X coincides with the smallest non-negative number n such that $\text{e-dim}(X) \leq K(G, n)$ where $K(G, n)$ is the Eilenberg-MacLane complex of G (this is a CW-complex having all homotopy groups trivial except for the n -th homotopy group $\pi_n(K(G, n))$ which is isomorphic to G). If no such n exists, then we put $\dim_G(X) = \infty$. It is known that $\dim_G(X) \leq \dim(X)$ for each abelian group G and $\dim(X) = \dim_{\mathbb{Z}}(X)$ for any finite-dimensional compact space X . On the other hand, the famous Pontryagin surface Π_2 has covering dimension $\dim \Pi_2 = 2$ and cohomological dimension $\dim_G(\Pi_2) = 1$ for any 2-divisible abelian group G , see [5, 1.9]. A group G is called *2-divisible* if for each $x \in G$ there is $y \in G$ with $y^2 = x$. Surprisingly, but for any totally Archimedean field K the space $M(K(x, y))$ has the same pathological dimension properties:

Theorem 1.4. *For any totally Archimedean field K the space of \mathbb{R} -places $M(K(x, y))$ has integral cohomological dimension $\dim_{\mathbb{Z}} M(K(x, y)) = \dim M(K(x, y)) = 2$ and the cohomological dimension $\dim_G M(K(x, y)) = 1$ for any non-trivial 2-divisible Abelian group G .*

Theorems 1.3 and 1.4 will be proved in Section 3 after some preliminary work made in Section 2.

2. GRAPHOIDS AND SPACES OF \mathbb{R} -PLACES

In this section we shall discuss the interplay between spaces of \mathbb{R} -places and graphoids. The notion of a graphoid has topological nature and can be defined for any family \mathcal{F} of partial functions between topological spaces.

By a *partial function* between topological spaces X, Y we understand a continuous function $f : \text{dom}(f) \rightarrow Y$ defined on a subspace $\text{dom}(f)$ of the space X . Its *graphoid* $\bar{\Gamma}(f)$ is the closure of its graph $\Gamma(f) = \{(x, f(x)) : x \in \text{dom}(f)\}$ in the Cartesian product $X \times Y$. The graphoid $\bar{\Gamma}(f)$ determines a multi-valued extension $\bar{f} : X \multimap Y$ of f whose graph $\Gamma(\bar{f}) = \{(x, y) \in X \times Y : y \in \bar{f}(x)\}$ coincides with the graphoid $\bar{\Gamma}(f)$ of f . The multivalued function $\bar{f} : X \multimap Y$ assigns to each point $x \in X$ the closed subset $\bar{f}(x) = \{y \in Y : (x, y) \in \bar{\Gamma}(f)\}$ of the space Y .

For a finite family \mathcal{F} of partial functions between topological spaces X, Y we define the graphoid $\bar{\Gamma}(\mathcal{F})$ of \mathcal{F} as the graphoid of the “vector” function

$$\mathcal{F} : \text{dom}(\mathcal{F}) \rightarrow Y^{\mathcal{F}}, \quad \mathcal{F} : x \mapsto (f(x))_{f \in \mathcal{F}},$$

defined on the subset $\text{dom}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \text{dom}(f)$.

For an arbitrary family \mathcal{F} of partial functions between X and Y we define its graphoid $\bar{\Gamma}(\mathcal{F})$ as the intersection

$$\bar{\Gamma}(\mathcal{F}) = \bigcap \{\text{pr}_{\mathcal{E}}^{-1}(\bar{\Gamma}(\mathcal{E})) : \mathcal{E} \subset \mathcal{F}, \quad |\mathcal{E}| < \infty\} \subset X \times Y^{\mathcal{F}}$$

where for $\mathcal{E} \subset \mathcal{F}$

$$\text{pr}_{\mathcal{E}} : X \times Y^{\mathcal{F}} \rightarrow X \times Y^{\mathcal{E}}, \quad \text{pr}_{\mathcal{E}} : (x, (y_f)_{f \in \mathcal{F}}) \mapsto (x, (y_f)_{f \in \mathcal{E}}),$$

denotes the natural projection.

The following lemma describing the structure of the graphoid $\bar{\Gamma}(\mathcal{F})$ easily follows from the definition of $\bar{\Gamma}(\mathcal{F})$.

Lemma 2.1. *The graphoid $\bar{\Gamma}(\mathcal{F})$ consists of all points $(x, (y_f)_{f \in \mathcal{F}}) \in X \times Y^{\mathcal{F}}$ such that for any finite subfamily $\mathcal{E} \subset \mathcal{F}$ and neighborhoods $O(x) \subset X$ and $O(y_f) \subset Y$ of the points x and y_f , $f \in \mathcal{E}$, there is a point $x' \in O(x) \cap \text{dom}(\mathcal{E})$ such that $f(x') \in O(y_f)$ for all $f \in \mathcal{E}$.*

Now we consider the graphoids in the context of rational functions of n variables. To shorten notation, we shall denote the n -tuple (x_1, \dots, x_n) by \vec{x} . So, $K(\vec{x})$ will denote the field $K(x_1, \dots, x_n)$ of rational functions of n variables with coefficients in a field K .

Observe that each rational function $f \in \mathbb{R}(\vec{x})$, written as an irreducible fraction $f = \frac{p}{q}$ of two polynomials $p, q \in \mathbb{R}(\vec{x})$, can be thought as a partial function $f : \text{dom}(f) \rightarrow \mathbb{R}$ defined on the open dense subset $\text{dom}(f) = \mathbb{R}^n \setminus (p^{-1}(0) \cap q^{-1}(0))$ of the n -dimensional torus \mathbb{R}^n .

Now we see that any family of rational functions $\mathcal{F} \subset \mathbb{R}(\vec{x})$ can be considered as a family of partial functions whose graphoid $\bar{\Gamma}(\mathcal{F}) \subset \bar{\mathbb{R}}^n \times \bar{\mathbb{R}}^{\mathcal{F}}$ is a well-defined closed subset of the compact Hausdorff space $\bar{\mathbb{R}}^n \times \bar{\mathbb{R}}^{\mathcal{F}}$.

Observe that for any finite subfamily $\mathcal{F} \subset \mathbb{R}(\vec{x})$ the subset $\text{dom}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \text{dom}(f)$ is open and dense in \mathbb{R}^n . Thus the graphoid $\bar{\Gamma}(\mathcal{F}) \subset \bar{\mathbb{R}}^n \times \bar{\mathbb{R}}^{\mathcal{F}}$ projects surjectively onto the n -torus $\bar{\mathbb{R}}^n$. The same fact is true for any family $\mathcal{F} \subset \mathbb{R}(\vec{x})$: its graphoid $\bar{\Gamma}(\mathcal{F})$ projects surjectively onto the n -torus $\bar{\mathbb{R}}^n$.

It turns out that for a subfield $\mathcal{F} \subset \mathbb{R}(\vec{x})$, containing $\mathbb{Q}(\vec{x})$, the graphoid $\bar{\Gamma}(\mathcal{F})$ can be identified with a subspace of the space of \mathbb{R} -places $M(\mathcal{F})$.

Theorem 2.2. *Let $\mathcal{F} \supset \mathbb{Q}(\vec{x})$ be a subfield of the field $\mathbb{R}(\vec{x})$.*

- (1) *Each point $\gamma = (\vec{a}, (y_f)_{f \in \mathcal{F}})$ of the graphoid $\bar{\Gamma}(\mathcal{F}) \subset \bar{\mathbb{R}}^n \times \bar{\mathbb{R}}^{\mathcal{F}}$ determines an \mathbb{R} -place*

$$\delta_\gamma : \mathcal{F} \rightarrow \bar{\mathbb{R}}, \quad \delta_\gamma : f \mapsto y_f.$$

To each rational function $f \in \mathcal{F}$ this \mathbb{R} -place assigns a point $\delta_\gamma(f) \in \bar{f}(\vec{a})$ where $\bar{f} : \bar{\mathbb{R}}^n \multimap \bar{\mathbb{R}}$ is the multivalued extension of f whose graph $\Gamma(f)$ coincides with the graphoid $\bar{\Gamma}(f)$ of f .

- (2) *The map*

$$\delta : \bar{\Gamma}(\mathcal{F}) \rightarrow M(\mathcal{F}), \quad \delta : \gamma \mapsto \delta_\gamma,$$

is a topological embedding.

- (3) *If $\mathcal{F} = \mathbb{K}(\vec{x})$ for some subfield $\mathbb{K} \subset \mathbb{R}$, then*

$$\delta(\bar{\Gamma}(\mathcal{F})) = \{\chi \in M(\mathcal{F}) : \chi|_{\mathbb{K}} = \text{id}\}.$$

Proof. 1. Fix a point $\gamma = (\vec{a}, (b_f)_{f \in \mathcal{F}}) \in \bar{\Gamma}(\mathcal{F}) \subset \bar{\mathbb{R}}^n \times \bar{\mathbb{R}}^{\mathcal{F}}$ and consider the function $\delta_\gamma : \mathcal{F} \rightarrow \bar{\mathbb{R}}$, $\delta_\gamma : f \mapsto b_f$.

Given any rational function $f \in \mathcal{F}$ consider its graphoid $\bar{\Gamma}(f)$, which is equal to the closure of its graph $\{(\vec{x}, f(\vec{x})) : \vec{x} \in \text{dom}(f)\}$ in $\bar{\mathbb{R}}^n \times \bar{\mathbb{R}}$. Next, consider the projection

$$\text{pr}_f : \bar{\mathbb{R}}^n \times \bar{\mathbb{R}}^{\mathcal{F}} \rightarrow \bar{\mathbb{R}}^n \times \bar{\mathbb{R}}, \quad \text{pr}_f : (\vec{x}, (y_f)_{f \in \mathcal{F}}) \mapsto (\vec{x}, y_f)$$

and observe that $\text{pr}_f(\gamma) = (\vec{a}, b_f) \in \bar{\Gamma}(f) = \Gamma(\bar{f})$ by the definition of the graphoid $\bar{\Gamma}(\mathcal{F})$. Consequently, $\delta_\gamma(f) = b_f \in \bar{f}(\vec{a})$.

In particular, $\delta_\gamma(x_i) \in \bar{x}_i(\vec{a}) = a_i$ for all $i \leq n$. Here a_i denotes the i -th coordinate of the vector $\vec{a} = (a_1, \dots, a_n)$. Also for any constant function $c \in \mathcal{F}$ we get $\delta_\gamma(c) = \bar{c}(\vec{a}) = c$. In particular, $\delta_\gamma(0) = 0$ and $\delta_\gamma(1) = 1$.

To show that δ_γ is an \mathbb{R} -place on the field \mathcal{F} , it remains to check that $\delta_\gamma(f + g) \in \delta_\gamma(f) \oplus \delta_\gamma(g)$ and $\delta_\gamma(f \cdot g) = \delta_\gamma(f) \odot \delta_\gamma(g)$ for any rational functions $f, g \in \mathcal{F}$. Consider the finite subfamily $\mathcal{E} = \{f, g, f + g, f \cdot g\} \subset \mathcal{F}$, its graph

$$\Gamma(\mathcal{E}) = \{(\vec{x}, (y_e)_{e \in \mathcal{E}}) \in \text{dom}(\mathcal{E}) \times \mathbb{R}^{\mathcal{E}} : \forall e \in \mathcal{E} \ y_e = e(\vec{x})\}$$

and its graphoid $\bar{\Gamma}(\mathcal{E}) = \overline{\Gamma(\mathcal{E})} \subset \bar{\mathbb{R}}^n \times \bar{\mathbb{R}}^{\mathcal{E}}$. Observe that for any point $(\vec{x}, (y_e)_{e \in \mathcal{E}}) \in \Gamma(\mathcal{E})$ we get

$$y_{f+g} = (f + g)(\vec{x}) = f(\vec{x}) + g(\vec{x}) = y_f + y_g$$

and similarly $y_{f \cdot g} = y_f \cdot y_g$.

Consequently, $\Gamma(\mathcal{E}) \subset \mathbb{R}^n \times Y$ where $Y = \{(y_e)_{e \in \mathcal{E}} \in \mathbb{R}^{\mathcal{E}} : y_{f+g} = y_f + y_g, y_{f \cdot g} = y_f \cdot y_g\}$. Observe that the closure of the set Y in $\bar{\mathbb{R}}^{\mathcal{E}}$ coincides with the subset

$$\bar{Y} = \{(y_e)_{e \in \mathcal{E}} \in \bar{\mathbb{R}}^{\mathcal{E}} : y_{f+g} \in y_f \oplus y_g, y_{f \cdot g} = y_f \odot y_g\}.$$

Consequently, $\text{pr}_{\mathcal{E}}(\bar{\Gamma}(\mathcal{F})) \subset \bar{\Gamma}(\mathcal{E}) \subset \bar{\mathbb{R}}^n \times \bar{Y}$ which implies the desired inclusions

$$\delta_\gamma(f + g) = b_{f+g} \in b_f \oplus b_g = \delta_\gamma(f) \oplus \delta_\gamma(g)$$

and

$$\delta_\gamma(f \cdot g) = b_{f \cdot g} \in b_f \odot b_g = \delta_\gamma(f) \odot \delta_\gamma(g).$$

2. It is easy to see that the map $\delta : \bar{\Gamma}(\mathcal{F}) \rightarrow M(\mathcal{F})$, $\delta : \gamma \mapsto \delta_\gamma$, is continuous. Let us show that it is injective. Take two distinct points $\gamma = (\vec{a}, (b_f)_{f \in \mathcal{F}})$ and $\gamma' = (\vec{a}', (b'_f)_{f \in \mathcal{F}})$ in the graphoid $\bar{\Gamma}(\mathcal{F})$. Then either $b_f \neq b'_f$ for some $f \in \mathcal{F}$ or $a_i \neq a'_i$ for some $i \leq n$.

If $b_f \neq b'_f$ for some f , then $\delta_\gamma(f) = b_f \neq b'_f = \delta_{\gamma'}(f)$ and hence $\delta_\gamma \neq \delta_{\gamma'}$. If $a_i \neq a'_i$ for some $i \leq n$, then for the monomial x_i , we get $\delta_\gamma(x_i) = x_i(\vec{a}) = a_i \neq a'_i = x_i(\vec{a}') = \delta_{\gamma'}(x_i)$ and again $\delta_\gamma \neq \delta_{\gamma'}$.

Therefore, the continuous map $\delta : \bar{\Gamma}(\mathcal{F}) \rightarrow M(\mathcal{F})$ is injective. Since the space $\bar{\Gamma}(\mathcal{F})$ is compact and $M(\mathcal{F})$ is Hausdorff, the map δ is a topological embedding.

3. Assume that $\mathcal{F} = \mathbb{K}(\vec{x})$ for some subfield \mathbb{K} of \mathbb{R} . Then inclusion $\delta(\bar{\Gamma}(\mathcal{F})) \subset \{\chi \in M(\mathcal{F}) : \chi|_{\mathbb{K}} = \text{id}\}$ follows from the statement (1). To prove the reverse inclusion we shall apply the Tarski-Seidenberg Transfer Principle [15]. This Principle says that for two real closed extensions R_1, R_2 of an ordered field K , a finite system of inequalities between polynomials with coefficients in K has a solution in R_1 if and only if it has a solution in the field R_2 .

Fix an \mathbb{R} -place $\chi : \mathcal{F} \rightarrow \bar{\mathbb{R}}$ such that $\chi|_{\mathbb{K}} = \text{id}$. By [3], the \mathbb{R} -place χ is induced by some total ordering P of the field \mathcal{F} . Taking into account that $\chi|_{\mathbb{K}} = \text{id}$ is the identity \mathbb{R} -place on the field \mathbb{K} , we conclude that the orders on \mathbb{K} induced from the ordered fields (\mathcal{F}, P) and $(\mathbb{R}, \mathbb{R}_+)$ coincide. Let $\hat{\mathbb{K}}$ be the relative algebraic closure of \mathbb{K} in the real closed field \mathbb{R} and $\hat{\mathcal{F}}$ be a real closure of the ordered field (\mathcal{F}, P) . The Uniqueness Theorem [13, XI.§2] for real closures guarantees that $\hat{\mathbb{K}}$ can be identified with the real closure of \mathbb{K} in the field $\hat{\mathcal{F}}$. By Theorem 6 of [12], the \mathbb{R} -place χ extends to a unique \mathbb{R} -place $\hat{\chi} : \hat{\mathcal{F}} \rightarrow \bar{\mathbb{R}}$. The \mathbb{R} -place $\hat{\chi}|_{\hat{\mathbb{K}}}$, being a unique \mathbb{R} -place on the real closed field $\hat{\mathbb{K}}$, coincides with the identity \mathbb{R} -place $\text{id} : \hat{\mathbb{K}} \rightarrow \bar{\mathbb{R}}$.

For every $i \leq n$ let $a_i = \chi(x_i)$, $\vec{a} = (a_1, \dots, a_n)$, and $b_f = \chi(f)$ for $f \in \mathcal{F}$. The inclusion $\chi \in \delta(\bar{\Gamma}(\mathcal{F}))$ will be proved as soon as we check that the point $\gamma = (\vec{a}, (b_f)_{f \in \mathcal{F}}) \in \bar{\mathbb{R}}^n \times \bar{\mathbb{R}}^{\mathcal{F}}$ belongs to the graphoid $\bar{\Gamma}(\mathcal{F})$. This will follow from Lemma 2.1 as soon as for any finite subfamily $\mathcal{E} \subset \mathcal{F}$, a neighborhood $O(\vec{a}) \subset \bar{\mathbb{R}}^n$ of the point $\vec{a} = (a_1, \dots, a_n)$ and neighborhoods $O(b_f) \subset \bar{\mathbb{R}}$ of the points b_f , $f \in \mathcal{E}$, we find a vector $\vec{z} = (z_1, \dots, z_n) \in O(\vec{a}) \cap \text{dom}(\mathcal{E})$ such that $f(\vec{z}) \in O(b_f)$ for all $f \in \mathcal{E}$.

We lose no generality assuming that $\{x_1, \dots, x_n\} \subset \mathcal{E}$ and $\prod_{i=1}^n O(\chi(x_i)) \subset O(\vec{a})$.

Also we can assume that for each function $f \in \mathcal{E}$ the neighborhood $O(b_f)$ is of the form

- $] \alpha_f, \beta_f[$ for some rational numbers $\alpha_f < \beta_f$ if $b_f \in \mathbb{R}$, and
- $\bar{\mathbb{R}} \setminus [\alpha_f, \beta_f]$ for some rational numbers $\alpha_f < \beta_f$ if $b_f = \infty$.

Write each rational function $f \in \mathcal{E}$ as an irreducible fraction $f = \frac{p_f}{q_f}$ of two polynomials $p_f, q_f \in \mathbb{K}(\vec{x})$. Replacing the polynomials p_f and q_f by $-p_f$ and $-q_f$, if necessary, we can assume that $q_f > 0$ in the ordered field $\hat{\mathcal{F}}$.

Write the finite set \mathcal{E} as the union $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_0 \cup \mathcal{E}_+$ where

$$\begin{aligned} \mathcal{E}_0 &= \{f \in \mathcal{E} : \chi(f) \in \mathbb{R}\}, \\ \mathcal{E}_- &= \{f \in \mathcal{E} : \chi(f) = \infty, f < 0 \text{ in } \hat{\mathcal{F}}\}, \\ \mathcal{E}_+ &= \{f \in \mathcal{E} : \chi(f) = \infty, f > 0 \text{ in } \hat{\mathcal{F}}\}. \end{aligned}$$

To each $f \in \mathcal{E}_0$ we shall assign a system of two polynomial inequalities that has a solution in the field $\hat{\mathcal{F}}$. Observe that the inclusion $b_f \in O(b_f) =] \alpha_f, \beta_f[$ implies that $\alpha_f < \chi(\frac{p_f}{q_f}) = \hat{\chi}(\frac{p_f}{q_f}) < \beta_f$. Since the \mathbb{R} -place $\hat{\chi}$ is generated by the total order of the real closed field $\hat{\mathcal{F}}$, these inequalities are equivalent to the inequalities $\alpha_f < \frac{p_f}{q_f} < \beta_f$ holding in the ordered field $\hat{\mathcal{F}}$. Since $q_f > 0$, the latter inequalities are equivalent to $\alpha_f q_f < p_f < \beta_f q_f$. It follows that the vector $\vec{x} = (x_1, \dots, x_n) \in \hat{\mathcal{F}}^n$ is a solution of the system

$$\alpha_f q_f(\vec{x}) < p_f(\vec{x}) < \beta_f q_f(\vec{x})$$

in the real closed field $\hat{\mathcal{F}}$.

Next, consider the case of a function $f \in \mathcal{E}_+$. Since $\hat{\chi}(\frac{p_f}{q_f}) = \chi(f) = \infty$ and $f > 0$, we get $\beta_f q_f < p_f$ in $\hat{\mathcal{F}}$ and hence the inequality

$$\beta_f q_f(\vec{x}) < p_f(\vec{x})$$

has solution $\vec{x} = (x_1, \dots, x_n)$ in $\hat{\mathcal{F}}$. By the same reason, for every $f \in \mathcal{E}_-$ the inequality

$$p_f(\vec{x}) < \alpha_f q_f(\vec{x})$$

has solution in $\hat{\mathcal{F}}$.

Therefore, the system of the inequalities

$$\begin{cases} q_f(\vec{x}) > 0 & \text{for all } f \in \mathcal{E} \\ \alpha_f q_f(\vec{x}) < p_f(\vec{x}) < \beta_f q_f(\vec{x}) & \text{for all } f \in \mathcal{E}_0 \\ \beta_f q_f(\vec{x}) < p_f(\vec{x}) & \text{for all } f \in \mathcal{E}_+ \\ p_f(\vec{x}) < \alpha_f q_f(\vec{x}) & \text{for all } f \in \mathcal{E}_- \end{cases}$$

has solution $\vec{x} = (x_1, \dots, x_n)$ in the real closed field $\hat{\mathcal{F}}$. By the Tarski-Seidenberg Transfer Principle [15, 11.2.2], this system has a solution in the real closed field $\mathbb{K} \subset \mathbb{R}$. Using the continuity of the polynomials p_f, q_f , $f \in \mathcal{E}$, we can find a solution \vec{z} of this system in the dense subset $(\mathbb{K} \cap \text{dom}(\mathcal{E}))^n$ of \mathbb{K}^n . The choice of the inequalities from the system guarantees that $\vec{z} \in \prod_{i=1}^n O(\chi(x_i)) \cap \text{dom}(\mathcal{E})^n \subset O(a) \cap \text{dom}(\mathcal{E})^n$ and $f(\vec{z}) = \frac{p_f(\vec{z})}{q_f(\vec{z})} \in O(b_f)$ for all $f \in \mathcal{E}$. \square

Theorem 2.2 will help us to analyze the structure of certain fibers of the restriction operator $\rho_K : M(K(\vec{x})) \rightarrow M(K)$, $\rho_K : \chi \mapsto \chi|_K$.

Proposition 2.3. *Take any field K with an injective \mathbb{R} -place $\varphi : K \rightarrow \mathbb{R}$. Then the fiber $\rho_K^{-1}(\varphi) \subset M(K(\vec{x}))$ can be identified with the graphoid $\bar{\Gamma}(\mathcal{F})$, where $\mathcal{F} = \mathbb{K}(\vec{x})$ for $\mathbb{K} = \varphi(K) \subset \mathbb{R}$.*

Proof. The \mathbb{R} -place $\varphi : K \rightarrow \mathbb{R}$, being injective, is an isomorphism of the fields K and \mathbb{K} . This isomorphism extends to a unique isomorphism $\Phi : K(\vec{x}) \rightarrow \mathbb{K}(\vec{x})$ such that $\Phi(x_i) = x_i$ for all $i \leq n$, where $\vec{x} = (x_1, \dots, x_n)$.

The isomorphism $\varphi : K \rightarrow \mathbb{K}$ induces a homeomorphism $M\varphi : M(\mathbb{K}) \rightarrow M(K)$ which assigns to each \mathbb{R} -place $\chi : \mathbb{K} \rightarrow \mathbb{R}$ the \mathbb{R} -place $\chi \circ \varphi : K \rightarrow \mathbb{R}$. In the same way the isomorphism Φ induces a homeomorphism $M\Phi : M(\mathbb{K}(\vec{x})) \rightarrow M(K(\vec{x}))$. Now look at the commutative diagram

$$\begin{array}{ccccc} M(K(\vec{x})) & \xleftarrow{M\Phi} & M(\mathbb{K}(\vec{x})) & & \\ \rho_K \downarrow & & \downarrow \rho_{\mathbb{K}} & \swarrow \delta & \\ M(K) & \xleftarrow{M\varphi} & M(\mathbb{K}) & & \bar{\Gamma}(\mathbb{K}(\vec{x})) \\ \uparrow & & \uparrow & \swarrow & \\ \{\varphi\} & \xleftarrow{\quad} & \{\text{id}\} & & \end{array}$$

Here $\delta : \bar{\Gamma}(\mathbb{K}(\vec{x})) \rightarrow M(\mathbb{K}(\vec{x}))$ is the embedding defined in Theorem 2.2, which implies that $\rho_{\mathbb{K}}^{-1}(\text{id}) = \delta(\bar{\Gamma}(\mathbb{K}(\vec{x})))$. Since the maps $M\varphi$ and $M\Phi$ are homeomorphisms, we conclude that the composition $M\Phi \circ \delta$ maps homeomorphically the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ onto the fiber $\rho_K^{-1}(\varphi)$. \square

3. EXTENSION DIMENSION OF THE SPACE $M(K(\vec{x}))$

In this section we shall evaluate the extension dimension of the space of \mathbb{R} -places $M(K(\vec{x}))$ of the field $K(\vec{x}) = K(x_1, \dots, x_n)$ of rational functions of n variables with coefficients in a field K .

We shall say that a topological space Y is an *absolute neighborhood extensor for compacta* (briefly, an ANE) if each continuous map $f : B \rightarrow Y$ defined on a closed subspace B of a compact Hausdorff space X can be extended to a continuous map $\bar{f} : A \rightarrow Y$ defined on a neighborhood A of B in X .

We recall that a topological space X has extension dimension $\text{e-dim } X \leq Y$ if each continuous map $f : B \rightarrow Y$ defined on a closed subspace B of X admits a continuous extension $\bar{f} : X \rightarrow Y$.

Theorem 3.1. *For a (totally Archimedean) field K the space of \mathbb{R} -places of the field $K(\vec{x})$ has extension dimension $\text{e-dim } M(K(\vec{x})) \leq Y$ for some ANE-space Y (if and) only if for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field K the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ has extension dimension $\text{e-dim } \bar{\Gamma}(\mathbb{K}(\vec{x})) \leq Y$.*

Proof. To prove the “only if” part, assume that $\text{e-dim } M(K(\vec{x})) \leq Y$ for some space Y . Given any subfield $\mathbb{K} \subset \mathbb{R}$, isomorphic to K , we need to check that $\text{e-dim } \bar{\Gamma}(\mathbb{K}(\vec{x})) \leq Y$. Fix any isomorphism $\varphi : K \rightarrow \mathbb{K}$ and observe that it is an injective \mathbb{R} -place on K . By Proposition 2.3, the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ of the function family $\mathbb{K}(\vec{x})$ is homeomorphic to a subspace of the space $M(K(\vec{x}))$. Because of that $\text{e-dim } M(K(\vec{x})) \leq Y$ implies $\text{e-dim } \bar{\Gamma}(\mathbb{K}(\vec{x})) \leq Y$.

The “if” part holds under the assumption that the field K is totally Archimedean and the space Y is an ANE. Assume that for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field K the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ has extension dimension $\text{e-dim } \bar{\Gamma}(\mathbb{K}(\vec{x})) \leq Y$. Since the field K is totally Archimedean, each \mathbb{R} -place $\chi : K \rightarrow \bar{\mathbb{R}}$ is injective, has image in \mathbb{R} and is generated by a unique total order on X (defined as $x < y$ iff $\chi(x) < \chi(y)$). This means that the quotient map $\lambda : \mathcal{X}(K) \rightarrow M(K)$ is injective and hence is a homeomorphism. Since the space $\mathcal{X}(K)$ is zero-dimensional, so is the space $M(K) = M_A(X)$.

Now consider the restriction operator $\rho_K : M(K(\vec{x})) \rightarrow M(K)$, $\rho_K : \chi \mapsto \chi|_K$. By Proposition 2.3, for each \mathbb{R} -place $\varphi \in M_A(K) = M(K)$ the fiber $\rho_K^{-1}(\varphi)$ is homeomorphic to the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ of the family $\mathbb{K}(\vec{x})$ of rational functions of n variables with coefficients in the subfield $\mathbb{K} = \varphi(K)$ of \mathbb{R} . Our assumption on the extension dimension of $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ implies that $\text{e-dim } \rho_K^{-1}(\varphi) \leq Y$. The following lemma implies that $\text{e-dim } M(K(\vec{x})) \leq Y$. \square

Lemma 3.2. *Let $\rho : X \rightarrow Z$ be a continuous map from a compact Hausdorff space X onto a zero-dimensional compact Hausdorff space Z . The space X has extension dimension $\text{e-dim } X \leq Y$ for some ANE-space Y if and only if for each $z \in Z$ the fiber $\rho^{-1}(z)$ has extension dimension $\text{e-dim } \rho^{-1}(z) \leq Y$.*

Proof. The “only if” trivially follows from the definition of extension dimension. To prove the “if” part, assume that each fiber of ρ has extension dimension $\leq Y$. To prove that $\text{e-dim } X \leq Y$, fix a continuous map $f : B \rightarrow Y$ defined on a closed subspace B of X . For each point $z \in Z$ consider the fiber $X_z = \rho^{-1}(z) \subset X$ of the map ρ . Since $\text{e-dim } X_z \leq Y$, the map $f|_{B \cap X_z}$ admits a continuous extension $\tilde{f}_z : X_z \rightarrow Y$. Consider the map $\tilde{f}_z : X_z \cup B \rightarrow Y$ defined by $\tilde{f}_z|_{X_z} = \tilde{f}_z$ and $\tilde{f}_z|_B = f$. Since Y is an ANE-space, the map \tilde{f}_z admits a continuous extension $\bar{f}_z : A_z \rightarrow Y$ defined on an open neighborhood A_z of $X_z \cup B$ in X . Since the space X is compact and Z is Hausdorff, the map ρ is closed. Consequently, the set $f(X \setminus A_z)$ is closed in Z and its complement $O_z = Z \setminus f(X \setminus A_z)$ is an open neighborhood of z in Z . Since the space Z is compact and zero-dimensional, the open cover $\{O_z : z \in Z\}$ of Z can be refined by a finite disjoint open cover \mathcal{U} . For every set $U \in \mathcal{U}$ choose a point $z \in Z$ with $U \subset O_z$ and put $\bar{f}_U = \bar{f}_z|_{\rho^{-1}(U)}$. It follows that the map \bar{f}_U is a continuous extension of the map $f|_{B \cap \rho^{-1}(U)}$. Then the maps \bar{f}_U , $U \in \mathcal{U}$, compose a required continuous extension $\bar{f} = \bigcup_{U \in \mathcal{U}} \bar{f}_U : X \rightarrow Y$ of the map f . \square

By Hurewicz-Wallman Theorem [9, 1.9.3], a compact Hausdorff space X has covering topological dimension $\dim X \leq d$ for some $d \in \omega$ if and only if $\text{e-dim } X \leq S^d$ where S^d stands for the d -dimensional sphere. Because of that Theorem 3.1 implies:

Corollary 3.3. *For a (totally Archimedean) field K the space of \mathbb{R} -places of the field $K(\vec{x})$ has dimension $\dim M(K(\vec{x})) \leq d$ for some $d \in \omega$ (if and) only if for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field K the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ has dimension $\dim \bar{\Gamma}(\mathbb{K}(\vec{x})) \leq d$.*

Let us recall [5] for an Abelian group G a compact Hausdorff space X has cohomological dimension $\dim_G X \leq d$ for some $d \in \omega$ if and only if $\text{e-dim } X \leq K(G, d)$ (see Introduction). This fact combined with Theorem 3.1 implies:

Corollary 3.4. *For a (totally Archimedean) field K the space of \mathbb{R} -places of the field $K(\vec{x})$ has dimension $\dim_G M(K(\vec{x})) \leq d$ for some $d \in \omega$ and some Abelian group G (if and) only if for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field K the graphoid $\bar{\Gamma}(\mathbb{K}(\vec{x}))$ has dimension $\dim_G \bar{\Gamma}(\mathbb{K}(\vec{x})) \leq d$.*

Now we see that Theorems 1.3 and 1.4 follows from Corollaries 3.3, 3.4 and the following deep result [1] about the (cohomological) dimension of the graphoids.

Theorem 3.5 (Banach-Potyatynyk). (1) *For any subfamily $\mathcal{F} \subset \mathbb{R}(x)$ the graphoid $\bar{\Gamma}(\mathcal{F})$ is homeomorphic to the extended real line $\bar{\mathbb{R}}$ and hence has dimension $\dim \bar{\Gamma}(\mathcal{F}) = 1$.*
 (2) *For any subfamily $\mathcal{F} \subset \mathbb{R}(x, y)$ the graphoid $\bar{\Gamma}(\mathcal{F})$ has dimensions $\dim \bar{\Gamma}(\mathcal{F}) = \dim_{\mathbb{Z}} \bar{\Gamma}(\mathcal{F}) = 2$.*
 (3) *For any subfamily $\mathcal{F} \subset \mathbb{R}(x, y)$ containing the rational functions $\frac{x-a}{y-b}$, $a, b \in \mathbb{Q}$, the graphoid $\bar{\Gamma}(\mathcal{F})$ has cohomological dimension $\dim_G \bar{\Gamma}(\mathcal{F}) = 1$ for any non-trivial 2-divisible Abelian group G .*

In light of Corollaries 3.3 and 3.4 the following problem arises naturally.

Problem 3.6. *Let $\mathbb{K}, \mathbb{F} \subset \mathbb{R}$ be two isomorphic copies of a (totally Archimedean) field K . Are the graphoids $\bar{\Gamma}(\mathbb{K}(x, y))$ and $\bar{\Gamma}(\mathbb{F}(x, y))$ homeomorphic?*

Remark 3.7. In light of this question it is interesting to remark that a totally Archimedean field can have distinct isomorphic copies in \mathbb{R} . A suitable example can be constructed as follows. Take the polynomial $f(x) = x^4 - 5x^2 + 2$. This polynomial is irreducible over \mathbb{Q} and has four real roots. The Galois group of f is the dihedral group with 8 elements, so the degree of the splitting field of f over \mathbb{Q} is 8. Therefore, for every root α of f there is another root β such that $\beta \notin \mathbb{Q}(\alpha)$. It follows that $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic, but not equal, totally Archimedean subfields of \mathbb{R} .

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